

Some new I_λ -lacunary statistical convergence

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Abstract

In this paper, we introduce some new I_λ -lacunary statistically convergent sequence spaces of order α defined by a Musielak-Orlicz function. We study some relations between I_λ -lacunary statistical convergence with lacunary I_λ -summable sequences. Moreover we also study about the I_λ -lacunary statistical convergence of sequences in topological groups and give some important inclusion theorems.

Keywords: Musielak-Orlicz function; Ideal Convergence; lacunary sequences; topological groups.

Introduction

Fast [3] introduced the concept of statistical convergence as the generalization of convergence of real sequences. Later on, it became as an important tool in the summability theory by many mathematicians. Also statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces and many more. For references one may see [[5], [6], [4], [18], [10]] and many more.

Kostyrko et al. [8] further extended the notion of statistical convergence to I -statistical convergence and studied some basic properties on it. A family of sets $I \subseteq 2^X$ (power sets of X) is said to be an ideal if I is additive i.e. $P, Q \in I \Rightarrow P \cup Q \in I$ and hereditary i.e. $P \in I, Q \subseteq P \Rightarrow Q \in I$, where X is any non empty set.

A lacunary sequence is defined as an increasing integer sequence $\theta = (j_r)$ such that $j_0 = 0$ and $h_r = j_r - j_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$.

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Note: Throughout this paper, the intervals determined by θ will be denoted by $J_r = (j_{r-1}, j_r]$ and the ratio $\frac{j_r}{j_{r-1}}$ will be defined by ϕ_r .

Preliminary Concepts

A sequence (x_i) of real numbers is said to be statistically convergent to Z if, for arbitrary $\xi > 0$, the set $K(\xi) = \{i \in \mathbb{N} : |x_i - Z| \geq \xi\}$ has natural density zero, i.e.,

$$\lim_i \frac{1}{i} \sum_{j=1}^i \chi_{K(\xi)}(j) = 0,$$

where $\chi_{K(\xi)}$ denotes the characteristic function of $K(\xi)$.

A sequence (x_i) of elements of \mathbb{R} is said to be I -convergent to $Z \in \mathbb{R}$ if for each $\xi > 0$,

$$\{i \in \mathbb{N} : |x_i - Z| \geq \xi\} \in I.$$

For any lacunary sequence $\theta = (j_r)$, the space N_θ defined as, (Freedman et al.[4])

$$N_\theta = \left\{ (x_j) : \lim_{r \rightarrow \infty} j_r^{-1} \sum_{j \in J_r} |x_j - Z| = 0, \text{ for some } Z \right\}.$$

By an Orlicz function, we mean a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

Musielak [15] defined the concept of Musielak-Orlicz function as $\mathcal{M} = (M_j)$. A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_j(u) : u \geq 0\}, k = 1, 2, \dots$$

which is called the complementary function of a Musielak-Orlicz function \mathcal{M} . The Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$\begin{aligned} t_{\mathcal{M}} &= \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\}, \\ h_{\mathcal{M}} &= \{x \in w : I_{\mathcal{M}}(cx) < \infty, \forall c > 0\}, \end{aligned}$$

where $I_{\mathcal{M}}$ is a convex modular defined by,

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_j(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

It is considered $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}} \left(\frac{x}{k} \right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{M}}(kx)) : k > 0 \right\}.$$

Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive integers. We denote Λ as the set of all non-decreasing sequence of positive integers. We consider a sequence $\{x_j\}_{j \in \mathbb{N}}$ which is said to be I_λ -lacunary statistically convergent of order α to Z , if for each $\gamma > 0$ and $\xi > 0$,

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{ j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{|x_j - Z|}{\rho(j)} \right) \geq \gamma \right\} \right| \geq \xi \right\} \in I.$$

We denote the class of all I_λ -lacunary statistically convergent sequences of order α defined by Musielak-Orlicz function as $S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$.

Some particular cases:

1. If $M_j(x) = M(x)$, for all $j \in \mathbb{N}$, then $S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$ is reduced to $S_{I_\lambda}^\alpha(M, \theta)$. Further, if $M_j(x) = x$, for all $j \in \mathbb{N}$, then $S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$ will be changed as $S_{I_\lambda}^\alpha(\theta)$.
2. If $\lambda_i = i$, for all $i \in \mathbb{N}$, then $S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$ will be reduced to $S_I^\alpha(\mathcal{M}, \theta)$.
3. If $\alpha = 1$, then α -density of any set reduces to the natural density of the set. So, the set $S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$ reduces to $S_{I_\lambda}(\mathcal{M}, \theta)$ for $\alpha = 1$.
4. If $\theta = (2^r)$ and $\alpha = 1$, then the sequence (x_j) is said to be I_λ -statistically convergent defined by Musielak-Orlicz function, i.e. $(x_j) \in S_{I_\lambda}(\mathcal{M})$.
5. If $M_j(x) = x$, $\theta = (2^r)$, $\lambda_j = j$, $\alpha = 1$, then I_λ -lacunary statistically convergence of order α defined by Musielak-Orlicz function reduces to I -statistical convergence.

In this article, we define the concept of I_λ -lacunary statistically convergent of order α defined by Musielak-Orlicz function and investigate some results on these sequences. Moreover, we study I_λ -lacunary statistically convergence of sequences in topological groups and give some important inclusion theorems.

Main Results

Theorem 0.1. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_i)$ be two sequences in Λ such that $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$ for fixed reals α and β . If $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$, then $S_{I_\mu}^\beta(\mathcal{M}, \theta) \subseteq S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$.

Proof. Suppose that $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$ and $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$. Since $I_i \subset J_i$, where $J_i = [i - \mu_i + 1, i]$, so for $\varepsilon > 0$, we can write,

$$\{j \in J_i : |x_j - Z| \geq \varepsilon\} \supset \{j \in I_i : |x_j - Z| \geq \varepsilon\}$$

which implies

$$\frac{1}{\mu_i^\beta} \left| \{j \in J_i : |x_j - Z| \geq \varepsilon\} \right| \geq \frac{\lambda_i^\alpha}{\mu_i^\beta} \cdot \frac{1}{\lambda_i^\alpha} \left| \{j \in I_i : |x_j - Z| \geq \varepsilon\} \right|,$$

for all $i \in \mathbb{N}$.

Assume that $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} = a$, so from the definition we get $\left\{ i \in \mathbb{N} : \frac{\lambda_i^\alpha}{\mu_i^\beta} < \frac{a}{2} \right\}$ is finite.

Now for $\delta > 0$,

$$\begin{aligned} \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\beta} \left| \{j \in J_i : |x_j - Z| \geq \varepsilon\} \right| \geq \delta \right\} &\subset \left\{ i \in \mathbb{N} : \frac{1}{\mu_i^\alpha} \left| \{j \in I_i : |x_j - Z| \geq \varepsilon\} \right| \geq \frac{a}{2} \delta \right\} \\ &\cup \left\{ i \in \mathbb{N} : \frac{\lambda_i^\alpha}{\mu_i^\beta} < \frac{a}{2} \right\}. \end{aligned}$$

Since I is admissible and (x_j) is I_μ -lacunary statistically convergent sequence of order β defined by a Musielak-Orlicz function, so by using the continuity of Musielak-Orlicz function, the set on the right hand side with the lacunary sequence $\theta = (h_i)$ belongs to I . This completes the proof. \square

Theorem 0.2. If $\lim_{i \rightarrow \infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, for $\lambda = (\lambda_i)$ and $\mu = (\mu_i)$ two sequences of Λ such that

$\lambda_i \leq \mu_i, \forall i \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$ for fixed α, β reals, then $S_{I_\lambda}^\alpha(\mathcal{M}, \theta) \subseteq S_{I_\mu}^\beta(\mathcal{M}, \theta)$.

Proof. Let (x_j) be I_λ -lacunary statistically convergent to Z of order α defined by the Musielak-Orlicz function \mathcal{M} . Also assume that $\lim_{i \rightarrow \infty} \frac{\mu_i}{\lambda_i^\beta} = 1$. So we can choose $m \in \mathbb{N}$

such that $\left| \frac{\mu_i}{\lambda_i^\beta} - 1 \right| < \frac{\delta}{2}, \forall i \geq m$.

Since $I_i \subset J_i$, for $\varepsilon > 0$, we may write,

$$\begin{aligned}
\frac{1}{\mu_i^\beta} \left| \left\{ j \in J_i : |x_j - Z| \geq \varepsilon \right\} \right| &= \frac{1}{\mu_i^\beta} \left| \left\{ i - \mu_i + 1 \leq j \leq i - \lambda_i : |x_j - Z| \geq \varepsilon \right\} \right| \\
&+ \frac{1}{\mu_i^\beta} \left| \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \right| \\
&\leq \frac{\mu_i - \lambda_i}{\mu_i^\beta} + \frac{1}{\mu_i^\beta} \left| \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \right| \\
&\leq \frac{\mu_i - \lambda_i^\beta}{\lambda_i^\beta} + \frac{1}{\mu_i^\beta} \left| \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \right| \\
&\leq \left(\frac{\mu_i}{\lambda_i^\beta} - 1 \right) + \frac{1}{\lambda_i^\alpha} \left| \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \right| \\
&= \frac{\delta}{2} + \frac{1}{\lambda_i^\alpha} \left| \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\{ i \in \mathbb{N} : \frac{1}{\mu_i^\beta} \left| \left\{ j \leq i : |x_j - Z| \geq \varepsilon \right\} \right| \geq \delta \right\} &\subset \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \right| \geq \frac{\delta}{2} \right\} \\
&\cup \{1, 2, 3, \dots, m\}.
\end{aligned}$$

Since (x_j) is I_λ -lacunary statistically convergent sequence of order α defined by a Musielak-Orlicz function \mathcal{M} and since I is admissible, so by using the continuity of Musielak-Orlicz function, we can say that the set on the right hand side with the lacunary sequence $\theta = (h_i)$ belongs to I and this proves that,

$$S_{I_\lambda}^\alpha(\mathcal{M}, \theta) \subseteq S_{I_\mu}^\beta(\mathcal{M}, \theta).$$

□

We define the lacunary I_λ -summable sequence of order α defined by Musielak-Orlicz function as:

$$w_{I_\lambda}^\alpha(\mathcal{M}, \theta) = \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left(j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \geq \varepsilon \right) \right\} \in I.$$

Theorem 0.3. Given for $\lambda = (\lambda_i), \mu = (\mu_i) \in \Lambda$. Suppose that $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}, 0 < \alpha \leq \beta \leq 1$. Then,

1. If $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$, then $w_\mu^\beta(\mathcal{M}, \theta) \subset w_\lambda^\alpha(\mathcal{M}, \theta)$.
2. If $\lim_{i \rightarrow \infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, then $\ell_\infty \cap w_\lambda^\alpha(\mathcal{M}, \theta) \subset w_\mu^\beta(\mathcal{M}, \theta)$.

Theorem 0.4. Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$, where $\lambda, \mu \in \Lambda$. Then, if $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$, and if (x_j) is lacunary I_μ -summable of order β defined by a Musielak-Orlicz function \mathcal{M} , then it is I_λ -lacunary statistically convergent of order α defined by the Musielak-Orlicz function \mathcal{M} .

Proof. For any $\varepsilon > 0$, we have,

$$\begin{aligned} \sum_{j \in I_i} |x_j - Z| &= \sum_{j \in I_i, |x_j - Z| \geq \varepsilon} |x_j - Z| + \sum_{j \in I_i, |x_j - Z| < \varepsilon} |x_j - Z| \\ &\geq \sum_{j \in I_i, |x_j - Z| \geq \varepsilon} |x_j - Z| + \sum_{j \in I_i, |x_j - Z| \geq \varepsilon} |x_j - Z| \\ &\geq \sum_{j \in I_i, |x_j - Z| \geq \varepsilon} |x_j - Z| \\ &\geq |\{j \in I_i : |x_j - Z| \geq \varepsilon\}| \cdot \varepsilon \end{aligned}$$

so,

$$\begin{aligned} \frac{1}{\mu_i^\beta} \sum_{j \in I_i} |x_j - Z| &\geq \frac{1}{\mu_i^\beta} |\{j \in I_i : |x_j - Z| \geq \varepsilon\}| \cdot \varepsilon \\ &\geq \frac{\lambda_i^\alpha}{\mu_i^\beta} \cdot \frac{1}{\lambda_i^\alpha} |\{j \in I_i : |x_j - Z| \geq \varepsilon\}| \cdot \varepsilon. \end{aligned}$$

If $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} = a$ then, $\left\{ i \in \mathbb{N} : \frac{\lambda_i^\alpha}{\mu_i^\beta} < \frac{a}{2} \right\}$ is finite. So, for $\delta > 0$, we get,

$$\begin{aligned} \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{ j \leq i : \sum_{j \in I_i} |x_j - Z| \geq \varepsilon \right\} \right| \geq \delta \right\} &\subset \left\{ i \in \mathbb{N} : \frac{1}{\mu_i^\beta} \left\{ j \in I_i : |x_j - Z| \geq \varepsilon \right\} \geq \frac{a}{2} \delta \right\} \\ &\cup \left\{ i \in \mathbb{N} : \frac{\lambda_i^\alpha}{\mu_i^\beta} < \frac{a}{2} \right\}. \end{aligned}$$

Since I is admissible and (x_j) is lacunary I_μ -summable sequence of order β defined by a Musielak-Orlicz function \mathcal{M} , using the continuity of the Musielak-Orlicz function \mathcal{M} and using the lacunary sequence $\theta = (h_i)$, we can conclude that, $w_{I_\mu}^\beta(\mathcal{M}, \theta) \subseteq S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$. \square

Theorem 0.5. Let $\lim_{i \rightarrow \infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, for fixed real numbers α and β such that $0 < \alpha \leq \beta \leq 1$ and $\lambda_i \leq \mu_i$, for all $i \in \mathbb{N}$, where $\lambda, \mu \in \Lambda$. Also let $\theta!$ be a refinement of θ . If a bounded sequence (x_j) is I_λ -lacunary statistically convergent sequence of order α defined by a Musielak-Orlicz function \mathcal{M} , then it is also lacunary I_μ -summable sequence of order β defined by the Musielak-Orlicz function \mathcal{M} . i.e. $S_{I_\lambda}^\alpha(\mathcal{M}, \theta) \subseteq w_{I_\mu}^\beta(\mathcal{M}, \theta!)$.

Proof. Suppose that (x_j) is I_λ -lacunary statistically convergent sequence of order α defined by a Musielak-Orlicz function \mathcal{M} .

Given that $\lim_{i \rightarrow \infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, so we can choose $m \in \mathbb{N}$ such that $\left| \frac{\mu_i}{\lambda_i^\beta} - 1 \right| < \frac{\delta}{2}, \forall i \geq m$.

Assume that there is a finite number of points $\theta! = (j_i^!)$ in the interval $I_i = (j_{i-1}, j_i]$. Let there exists exactly one point $j_i^!$ of $\theta!$ in the interval I_i , that is, $j_{i-1} = j_{p-1}^! < j_p^! < j_{p+1}^! = j_i$. Let

$$I_i^1 = (j_{i-1}, j_p], I_i^2 = (j_p, j_i], h_i^1 = j_p - j_{i-1}, h_i^2 = j_i - j_p.$$

$$\begin{aligned} \frac{1}{\mu_i^\beta} \left(h_i^{-1} \sum_{j \in I_i} |x_j - Z| \right) &\leq \frac{1}{\mu_i^\beta} \left((h_i^{-1} h_i^1) h_i^{-1} \sum_{j \in I_i^1} |x_j - Z| + (h_i^{-1} h_i^2) h_i^{-2} \sum_{j \in I_i^2} |x_j - Z| \right) \\ &\leq \left(\frac{\mu_i - \lambda_i}{\mu_i^\beta} \right) (h_i^{-1} h_i^1) h_i^{-1} M + \frac{1}{\mu_i^\beta} \left((h_i^{-1} h_i^2) h_i^{-2} \sum_{j \in I_i^2} |x_j - Z| \right) \\ &\leq \left(\frac{\mu_i - \lambda_i^\beta}{\lambda_i^\beta} \right) (h_i^{-1} h_i^1) h_i^{-1} M + \frac{1}{\mu_i^\beta} \left((h_i^{-1} h_i^2) h_i^{-2} \sum_{j \in I_i^2} |x_j - Z| \right) \\ &\leq \left(\frac{\mu_i}{\lambda_i^\beta} - 1 \right) (h_i^{-1} h_i^1) h_i^{-1} M + \frac{1}{\mu_i^\beta} \left((h_i^{-1} h_i^2) h_i^{-2} \sum_{j \in I_i^2, |x_j - Z| \geq \varepsilon} |x_j - Z| \right) \\ &\quad + \frac{1}{\mu_i^\beta} \left((h_i^{-1} h_i^2) h_i^{-2} \sum_{j \in I_i^2, |x_j - Z| < \varepsilon} |x_j - Z| \right) \\ &\leq \left(\frac{\mu_i}{\lambda_i^\beta} - 1 \right) (h_i^{-1} h_i^1) h_i^{-1} M + \frac{M}{\lambda_i^\alpha} \left| \left\{ j \in I_i : (h_i^{-1} h_i^2) h_i^{-2} |x_j - Z| \geq \varepsilon \right\} \right| + \varepsilon (h_i^{-1} h_i^2) h_i^{-2}, \forall i \in \mathbb{N} \\ &= \frac{\delta}{2} (h_i^{-1} h_i^1) h_i^{-1} M + \frac{M}{\lambda_i^\alpha} \left| \left\{ j \in I_i : (h_i^{-1} h_i^2) h_i^{-2} |x_j - Z| \geq \varepsilon \right\} \right| + \varepsilon (h_i^{-1} h_i^2) h_i^{-2}. \end{aligned}$$

Since $x \in w_{I_\mu}^\beta(\mathcal{M}, \theta!)$, so $0 < h_i^{-1} h_i^1 \leq 1$ and $0 < h_i^{-1} h_i^2 \leq 1$.

Hence, for $\delta > 0$,

$$\begin{aligned} \left\{ i \in \mathbb{N} : \frac{1}{\mu_i^\beta} \left(\frac{1}{h_i} \sum_{j \in I_i} |x_j - Z| \geq \varepsilon \right) \geq \delta \right\} &\subset \left\{ i \in \mathbb{N} : \frac{M}{\lambda_i^\alpha} \left| \left\{ j \in I_i : \frac{1}{h_i^2} |x_j - Z| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ &\cup \{1, 2, 3, \dots, m\}. \end{aligned}$$

Since (x_j) is I_λ -lacunary statistical convergent sequence of order α defined by the Musielak-Orlicz function \mathcal{M} and since I is admissible, so by using the continuity of

\mathcal{M} , we can say that the set on the right hand side belongs to I and this proves that,

$$S_{I_\lambda}^\alpha(\mathcal{M}, \theta) \subseteq w_{I_\mu}^\beta(\mathcal{M}, \theta!).$$

□

Corollary 0.1. Let $\lambda_{\leq \mu_i}$ for all $i \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. Let $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$, $\theta!$ be the refinement of θ . Also let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function where (M_i) is pointwise convergent. Then $w_{I_\mu}^\beta(\mathcal{M}, \theta!) \subset S_{I_\lambda}^\alpha(\mathcal{M}, \theta)$ iff $\lim_i M_i\left(\frac{v}{\rho^{(i)}}\right) > 0$, for some $v > 0$, $\rho^{(i)} > 0$.

Corollary 0.2. Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function and $\lim_{i \rightarrow \infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, for fixed numbers α and β such that $0 < \alpha \leq \beta \leq 1$. Then $S_{I_\lambda}^\alpha(\mathcal{M}, \theta) \subset w_{I_\mu}^\beta(\mathcal{M}, \theta)$ iff $\sup_v \sup_i \left(\frac{v}{\rho^{(i)}}\right)$.

I_λ -lacunary statistical convergence in topological group

We define I_λ -lacunary statistical convergent of order α in topological groups. Let Λ be the collection of all non-decreasing sequence of positive integers $\lambda = (\lambda_i)$. We consider a sequence $\{x_i\}_{i \in \mathbb{N}}$ in X which is said to be I_λ -lacunary statistically convergent of order α to Z if for each neighborhood V of 0 and each $\xi > 0$,

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{ j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \geq \xi \right\} \in I.$$

In this case, we write $x_j \rightarrow L(S_{I_\lambda}^\alpha(\mathcal{M}, \theta))$ and denote by $S_{I_\lambda}^\alpha(X, \mathcal{M}, \theta)$ the set of all I_λ -lacunary statistically convergent sequence of order α in X .

Theorem 0.6. If $\lambda = (\lambda_i), \mu = (\mu_i) \in \Lambda$ such that $\lambda_i \leq \mu_i$, for all $i \in \mathbb{N}$ with $\liminf_{i \rightarrow \infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$ and $0 < \alpha \leq \beta \leq 1$ for fixed reals α and β then $S_{I_\mu}^\beta(X, \mathcal{M}, \theta) \subset S_{I_\lambda}^\alpha(X, \mathcal{M}, \theta)$.

Proof. Let us take any neighborhood V of 0.

Then,

$$\begin{aligned} & \frac{1}{\mu_i^\beta} \left| \left\{ j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ & \geq \frac{1}{\mu_i^\beta} \left| \left\{ j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ & \geq \frac{\lambda_i^\alpha}{\mu_i^\beta} \frac{1}{\lambda_i^\alpha} \left| \left\{ j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right|. \end{aligned}$$

If $\lim_{i \rightarrow \infty} \inf \frac{\lambda_i^\alpha}{\mu_i^\beta} = b$, then the set $\left\{i \in \mathbb{N} : \frac{\lambda_i^\alpha}{\mu_i^\beta} < \frac{b}{2}\right\}$ is finite. Thus, for $\xi > 0$ and any neighborhood V of 0,

$$\begin{aligned} & \left\{i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \geq \xi \right\} \\ & \subset \left\{i \in \mathbb{N} : \frac{1}{\mu_i^\beta} \left| \left\{j \leq i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \geq \frac{b}{2}\xi \right\} \cup \left\{i \in \mathbb{N} : \frac{\lambda_i^\alpha}{\mu_i^\beta} < \frac{b}{2}\right\}. \end{aligned}$$

So, if $x_i \rightarrow L(S_{I_\mu}^\beta(\mathcal{M}, \theta))$, then the set on the right hand side belongs to I . This completes the proof. \square

Theorem 0.7. Let $0 < \alpha \leq \beta \leq 1$ for some fixed reals α and β and let $\lambda, \mu \in \Lambda$ be such that $\lambda_i \leq \mu_i$. If $\lim_i \frac{\mu_i}{\lambda_i^\alpha} = 1$. Then $S_{I_\lambda}^\alpha(X, \mathcal{M}, \theta) \subset S_{I_\mu}^\beta(X, \mathcal{M}, \theta)$.

Proof. Let $\xi > 0$ be given. Since $\lim_i \frac{\mu_i}{\lambda_i^\alpha} = 1$, we can choose $t \in \mathbb{N}$ such that,

$$\left| \frac{\mu_i}{\lambda_i^\beta} - 1 \right| < \frac{\xi}{2}, \text{ for all } i \geq t.$$

Let us take any neighborhood V of 0. Now observe that,

$$\begin{aligned} & \frac{1}{\mu_i^\beta} \left| \left\{j \leq i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ &= \frac{1}{\mu_i^\beta} \left| \left\{i - \mu_i + 1 < j < i - \lambda_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| + \frac{1}{\mu_i^\beta} \left| \left\{j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ &< \frac{\mu_i - \lambda_i}{\mu_i^\beta} + \frac{1}{\mu_i^\beta} \left| \left\{j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ &< \frac{\mu_i - \lambda_i^\beta}{\lambda_i^\beta} + \frac{1}{\mu_i^\beta} \left| \left\{j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ &< \left(\frac{\mu_i}{\lambda_i^\beta} - 1 \right) + \frac{1}{\lambda_i^\alpha} \left| \left\{j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \\ &< \frac{\xi}{2} + \frac{1}{\lambda_i^\alpha} \left| \left\{j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right|, \text{ for all } i \geq t. \end{aligned}$$

Hence, for $\xi > 0$ and any neighborhood V of 0,

$$\left\{ i \in \mathbb{N} : \frac{1}{\mu_i^\beta} \left| \left\{ j \leq i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \geq \xi \right\} \\ \subset \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{ j \in I_i : \frac{1}{h_i} \sum_j M_j \left(\frac{|x_j - Z|}{\rho^{(j)}} \right) \notin V \right\} \right| \geq \frac{\xi}{2} \right\} \cup \{1, 2, \dots, t\}.$$

If $x_i \rightarrow L(S_{I_\lambda}^\alpha(\mathcal{M}, \theta))$, then the set on the right hand side belongs to I and so the set on the left hand side also belongs to I . This shows that (x_i) is I_μ -lacunary statistically convergent of order β to Z . \square

Competing interests

The authors declare that they have no competing interests.

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